

# A NOTE ON SUCCESSIVE COEFFICIENTS OF CONVEX FUNCTIONS

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**ABSTRACT.** In this note, we investigate the supremum and the infimum of the functional  $|a_{n+1}| - |a_n|$  for functions, convex and analytic on the unit disk, of the form  $f(z) = z + a_2z^2 + a_3z^3 + \dots$ . We also consider the related problem to maximize the functional  $|a_{n+1} - a_n|$  for convex functions  $f$  with  $f''(0) = p$  for a prescribed  $p \in [0, 2]$ .

## 1. INTRODUCTION

Let  $\mathcal{S}$  be the class of normalized analytic univalent functions  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  on the unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ . We sometimes write  $a_n = a_n(f)$  to indicate the function  $f$ . The Bieberbach conjecture asserts that  $|a_n(f)| \leq n$  for  $f \in \mathcal{S}$  with equality holding only for the Koebe function  $K(z) = z/(1-z)^2 = z + 2z^2 + 3z^3 + \dots$  and its rotation  $e^{-i\theta}K(e^{i\theta}z)$ . It had been a long-standing problem in Geometric Function Theory and was finally proved by de Branges [1]. At least, soon after the conjecture was posed, it was recognized that  $|a_n(f)| \leq Cn$  holds for  $f \in \mathcal{S}$  with an absolute constant  $C \leq e = 2.718\dots$  (see, for example, [2, p. 37]). Therefore, it is somewhat surprising that the difference  $|a_{n+1}| - |a_n|$  is bounded for  $f \in \mathcal{S}$ . Indeed, Hayman proved in his 1963 paper [5] that

$$(1.1) \quad ||a_{n+1}| - |a_n|| \leq A, \quad n = 1, 2, 3, \dots,$$

for  $f(z) = z + a_2z^2 + \dots$  in  $\mathcal{S}$ , where  $A \geq 1$  is an absolute constant. Note that (1.1) implies that  $|a_n| \leq An$ ,  $n = 2, 3, \dots$ . Unfortunately, it is known that the constant  $A$  must be greater than 1. In fact, the sharp inequalities

$$-1 \leq |a_3| - |a_2| \leq \frac{3}{4} + e^{-\lambda_0}(2e^{-\lambda_0} - 1) = 1.02908\dots$$

hold for  $f(z) = z + a_2z^2 + \dots$  in  $\mathcal{S}$ , where  $\lambda_0 \approx 0.3574$  is the solution  $\lambda$  in  $(0, 1)$  to the equation  $4\lambda e^{-\lambda} = 1$  (see [2, Theorem 3.11]). Schaeffer and Spencer [10] showed even that for each  $n \geq 2$ , there is an *odd* univalent function  $h \in \mathcal{S}$  with real coefficients such that  $|a_{2n+1}(h)| > 1$ . (It is well known that  $|a_3| \leq 1$  for every odd univalent function  $h(z) = z + a_3z^3 + a_5z^5 + \dots$ , see [2, p. 104].) The problem to find the minimal value for the constant  $A$  in (1.1) is not solved yet. The best result so far is the estimate  $A < 3.61$  due to Grinspan [4].

A function  $f \in \mathcal{S}$  is called *starlike* (resp. *convex*) if the image  $f(\mathbb{D})$  is starlike with respect to the origin (resp. convex). The class of starlike functions is denoted by  $\mathcal{S}^*$  and

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the class of convex functions is denoted by  $\mathcal{K}$ . In 1978 Leung [6] proved that (1.1) holds with  $A = 1$  for  $f \in \mathcal{S}^*$  (see also [2, §5.10]).

**Theorem A** (Leung). *For every  $f \in \mathcal{S}^*$ , the following inequalities hold:*

$$-1 \leq |a_{n+1}| - |a_n| \leq 1, \quad n = 1, 2, 3, \dots$$

*For each  $n \geq 2$ , equality occurs in the left-hand side if and only if  $f$  is  $K_\phi$  or its rotation  $e^{-i\theta}K_\phi(e^{i\theta}z)$  with  $\phi = k\pi/n$  for some integer  $k$  with  $0 \leq k \leq n/2$ . Likewise, equality occurs in the right-hand side if and only if  $f$  is  $K_\phi$  or its rotation with  $\phi = k\pi/(n+1)$  for some integer  $k$  with  $1 \leq k \leq (n+1)/2$ . Here,*

$$K_\phi(z) = \frac{z}{1 - 2z \cos \phi + z^2} = \sum_{n=1}^{\infty} \frac{\sin n\phi}{\sin \phi} z^n.$$

Note that one of  $a_n$  and  $a_{n+1}$  vanishes when equality holds in the theorem. The reader may consult [2, §3.10, §5.9 and §5.10] for more information about the difference of successive coefficients. In this note, we will look for a counterpart of Theorem A for convex functions. Since the result seems to be asymmetric in this case, to clarify the assertion, we consider the two quantities

$$(1.2) \quad \mathcal{D}_n^+ = \sup_{f \in \mathcal{K}} (|a_{n+1}(f)| - |a_n(f)|) \quad \text{and} \quad \mathcal{D}_n^- = \sup_{f \in \mathcal{K}} (|a_n(f)| - |a_{n+1}(f)|)$$

for  $n = 1, 2, 3, \dots$ . Note that the suprema can be replaced by maxima in (1.2) because of compactness of the class  $\mathcal{K}$ . It is well known that the sharp inequalities  $|a_n| \leq 1$  hold for a convex function  $f(z) = z + a_2z^2 + a_3z^3 + \dots$ , so that one easily gets  $\mathcal{D}_n^+ \leq 1$  and  $\mathcal{D}_n^- \leq 1$ . When  $n = 1$ , we easily have  $\mathcal{D}_1^+ = 0$  and  $\mathcal{D}_1^- = 1$ . From now on, we thus assume that  $n \geq 2$ . We will show that  $\mathcal{D}_n^+$  and  $\mathcal{D}_n^-$  are much smaller than 1. Before presenting our main results, we recall a related result due to Robertson [9].

**Theorem B** (Robertson). *Let  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  be a convex function. Then for each  $n \geq 2$ , the following inequality holds:*

$$(1.3) \quad |a_{n+1} - a_n| \leq \frac{2n+1}{3} |a_2 - 1|.$$

*The factor  $(2n+1)/3$  cannot be replaced by any smaller number independent of  $f$ .*

The sharpness of the factor was confirmed by the fact that

$$\frac{a_{n+1}(L_\phi) - a_n(L_\phi)}{a_2(L_\phi) - 1} \rightarrow \frac{2n+1}{3}$$

as  $\phi \rightarrow 0$ , where  $L_\phi$  is the convex function given by

$$(1.4) \quad L_\phi(z) = \frac{1}{e^{i\phi} - e^{-i\phi}} \log \frac{1 - e^{-i\phi}z}{1 - e^{i\phi}z} = \sum_{n=1}^{\infty} \frac{\sin n\phi}{n \sin \phi} z^n$$

for  $\phi \in \mathbb{R}$ . Here, we should take a suitable limit when  $\sin \phi = 0$ . For instance,  $L_0(z) = \lim_{\phi \rightarrow 0} L_\phi(z) = z/(1-z) = z + z^2 + z^3 + \dots$ . Note the relation  $K_\phi(z) = zL'_\phi(z)$ ; namely,  $L_\phi$  is a natural counterpart of  $K_\phi$  for convex functions.

**Theorem 1.1.** *Let  $\mathcal{D}_n^+$  and  $\mathcal{D}_n^-$  be given in (1.2). Then the following hold.*

- (i)  $\mathcal{D}_n^+ = 1/(n+1)$  for  $n \geq 2$ , and the function  $L_{\pi/n}$  is extremal for  $\mathcal{D}_n^+$ .
- (ii)  $\mathcal{D}_2^- = 1/2$ , and  $L_{\pi/3}$  is extremal for  $\mathcal{D}_2^-$ .
- (iii)  $\mathcal{D}_3^- = 1/3$ , and  $L_{\pi/4}$  is extremal for  $\mathcal{D}_3^-$ .

In view of (ii) and (iii) in the theorem, one might expect that  $\mathcal{D}_n^- = 1/n$  and that the function  $L_{\pi/(n+1)}$  would be extremal for  $\mathcal{D}_n^-$  when  $n \geq 4$ , as well. It is, however, not true. We will, in fact, prove the following.

**Theorem 1.2.**  $1/n < \mathcal{D}_n^- < 2/(n+1)$  for each  $n \geq 4$ .

It is an open problem to find the value of  $\mathcal{D}_n^-$  for  $n \geq 4$ .

As the triangle inequality implies  $||a_n| - |a_{n+1}|| \leq |a_{n+1} - a_n|$ , one may think that the study of the functional  $|a_{n+1} - a_n|$  would be helpful to our problem. However, we immediately see that the sharp bound of  $|a_{n+1} - a_n|$  for convex functions is 2 as the function  $f(z) = z/(1+z)$  serves as an extremal one. On the other hand, it is indeed helpful to consider the functional  $|a_{n+1} - a_n|$  for refined subclasses of  $\mathcal{K}$ . For a given number  $p$  with  $0 \leq p \leq 2$ , let

$$\mathcal{K}(p) = \{f \in \mathcal{K}, f''(0) = p\}.$$

Note that the union  $\mathcal{K}^+ = \bigcup_{0 \leq p \leq 2} \mathcal{K}(p)$  is smaller than the whole class  $\mathcal{K}$ . This sort of refined subclasses of  $\mathcal{K}$  were considered, for instance, in [12]. On the other hand, each function  $f$  in  $\mathcal{K}$  is a suitable rotation of a function in  $\mathcal{K}(p)$  for  $p = |f''(0)|$ . Since the functional  $|a_{n+1}| - |a_n|$  is rotationally invariant, one can replace  $\mathcal{K}$  by  $\mathcal{K}^+$  in the definition (1.2) of  $\mathcal{D}_n^+$  and  $\mathcal{D}_n^-$ . We should note that  $|a_{n+1} - a_n|$  is not necessarily invariant under rotations. Therefore, it might be meaningful to consider the extremal problem maximizing  $|a_{n+1} - a_n|$  among the class  $\mathcal{K}(p)$  for a given  $p \in [0, 2]$ . Noting the relation  $p = 2a_2$  for  $f \in \mathcal{K}(p)$ , Robertson's theorem (Theorem B) implies

$$|a_3 - a_2| \leq \frac{5(2-p)}{6} \quad \text{and} \quad |a_4 - a_3| \leq \frac{7(2-p)}{6}$$

for  $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$  in  $\mathcal{K}(p)$ . These inequalities can be improved in the following way.

**Theorem 1.3.** Let  $0 \leq p \leq 2$ . Suppose that  $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$  is a function in  $\mathcal{K}(p)$ . Then the following sharp inequalities hold:

$$(1.5) \quad |a_3 - a_2| \leq \frac{(2p+1)(2-p)}{6}, \quad \text{and}$$

$$(1.6) \quad |a_4 - a_3| \leq \begin{cases} \frac{p^3 + 50p^2 - 64p + 64}{192}, & \text{when } 0 \leq p < \frac{8}{7}, \\ \frac{-3p^3 + 4p^2 + 6p - 4}{12}, & \text{when } \frac{8}{7} \leq p \leq 2. \end{cases}$$

Furthermore,

$$(1.7) \quad \sup_{f \in \mathcal{K}^+} |a_3(f) - a_2(f)| = \frac{25}{48} \approx 0.520833,$$

where the supremum is attained by  $L_\phi$  with  $\phi = \arccos[3/8]$ . Likewise,

$$(1.8) \quad \sup_{f \in \mathcal{K}^+} |a_4(f) - a_3(f)| = \frac{35\sqrt{70} - 49}{729} \approx 0.334473,$$

where the supremum is attained by  $L_\phi$  with  $\phi = \arccos[(4 + \sqrt{70})/18]$ .

We will prove this theorem in Section 3. In Section 4, we will prove Theorems 1.1 and 1.2. The next section will be devoted to preparations for the proofs.

## 2. PRELIMINARIES

Let  $\mathcal{P}$  denote the class of analytic functions  $P$  with positive real part on  $\mathbb{D}$  which has the form

$$P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n.$$

A member of  $\mathcal{P}$  is called a Carathéodory function. Some preliminary lemmas are needed for the proof of our results. The first one is known as Carathéodory's lemma (see [2, p. 41] for example).

**Lemma 2.1.** *For a function  $P \in \mathcal{P}$ , the sharp inequality  $|p_n| \leq 2$  holds for each  $n$ .*

The sharpness can be observed through the example  $P_0(z) = (1+z)/(1-z) = 1 + 2z + 2z^2 + \dots$ . We will use also the following result due to Carathéodory and Toeplitz (see [3] or [11]).

**Lemma 2.2** (Carathéodory-Toeplitz theorem). *Let  $P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$  be a formal power series with complex coefficients. Then  $P$  represents a Carathéodory function if and only if*

$$(2.1) \quad D_n = \begin{vmatrix} 2 & p_1 & p_2 & \cdots & p_n \\ p_{-1} & 2 & p_1 & \cdots & p_{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ p_{-n} & p_{-n+1} & p_{-n+2} & \cdots & 2 \end{vmatrix}$$

is non-negative for each  $n \geq 1$ , where  $p_{-j} = \overline{p_j}$  for  $j \geq 1$ . Moreover, if  $D_1 > 0, \dots, D_{k-1} > 0$  and if  $D_k = 0$ , then  $P(z)$  is of the following form:

$$(2.2) \quad P(z) = \sum_{j=1}^k \gamma_j \frac{1 + \varepsilon_j z}{1 - \varepsilon_j z}, \quad \gamma_j > 0, \quad |\varepsilon_j| = 1, \quad \varepsilon_j \neq \varepsilon_h \ (j \neq h).$$

Since  $P(0) = 1$ , the numbers  $\gamma_j$  must satisfy  $\gamma_1 + \dots + \gamma_k = 1$ . As a special case with  $k = 2$ , one can deduce the following useful assertion from Lemma 2.2.

**Lemma 2.3.** *Let  $P(z) = 1 + p_1 z + p_2 z^2 + \dots$  be a Carathéodory function with  $p_1 \in \mathbb{R}$  and  $p_2 = p_1^2 - 2$ . Then  $P$  must be of the form*

$$P(z) = \frac{1 - z^2}{1 - p_1 z + z^2}.$$

Moreover, the functions  $f \in \mathcal{S}^*$  and  $g \in \mathcal{K}$  determined by  $zf'(z)/f(z) = 1 + zg''(z)/g'(z) = P(z)$  have the forms  $f = K_\phi$  and  $g = L_\phi$ , where  $\phi = \arccos[p_1/2]$ .

*Proof.* We first note that  $p_1, p_2 \in [-2, 2]$  by Lemma 2.1. We next observe that  $D_2 = 2(p_2 - 2)(p_2 - p_1^2 + 2) = 0$  by assumption, where  $D_2$  is given in (2.1) with  $n = 2$ . When  $D_1 = 0$ , which is equivalent to the condition  $p_1 = \pm 2$ , the assertion holds clearly with  $\phi = 0$  or  $\pi$ . Thus, we may assume that  $D_1 = 4 - p_1^2 > 0$ . Lemma 2.2 now implies that  $P$  has the form

$$P(z) = \gamma_1 \frac{1 + \varepsilon_1 z}{1 - \varepsilon_1 z} + \gamma_2 \frac{1 + \varepsilon_2 z}{1 - \varepsilon_2 z} = 1 + 2(\gamma_1 \varepsilon_1 + \gamma_2 \varepsilon_2)z + 2(\gamma_1 \varepsilon_1^2 + \gamma_2 \varepsilon_2^2)z^2 + \cdots$$

for  $\gamma_j > 0$  and  $\varepsilon_j \in \partial\mathbb{D}$  ( $j = 1, 2$ ) with  $\gamma_1 + \gamma_2 = 1$  and  $\varepsilon_1 \neq \varepsilon_2$ . We write  $\varepsilon_j = e^{i\phi_j}$  for  $\phi_j \in (-\pi, \pi]$ . By comparing the coefficients of  $z$  and  $z^2$  in the above formula, we obtain the relations

$$(2.3) \quad \gamma_1 \varepsilon_1 + \gamma_2 \varepsilon_2 = \frac{p_1}{2},$$

$$(2.4) \quad \gamma_1 \varepsilon_1^2 + \gamma_2 \varepsilon_2^2 = \frac{p_2}{2} = \frac{p_1^2}{2} - 1.$$

If one of  $\varepsilon_1, \varepsilon_2$  is real, by (2.3), the other must be real, too. This occurs only when  $\varepsilon_1 = -\varepsilon_2 = \pm 1$  so that (2.4) implies  $\gamma_1 + \gamma_2 = p_1^2/2 - 1 < 1$ , which contradicts  $\gamma_1 + \gamma_2 = 1$ . Hence,  $\varepsilon_1$  and  $\varepsilon_2$  are both non-real; i.e.,  $\sin \phi_1 \sin \phi_2 \neq 0$ . Taking the imaginary part of (2.3) and (2.4), we have

$$\begin{pmatrix} \sin \phi_1 & \sin \phi_2 \\ \sin 2\phi_1 & \sin 2\phi_2 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since  $(\gamma_1, \gamma_2)$  is a non-zero vector, one has

$$\begin{vmatrix} \sin \phi_1 & \sin \phi_2 \\ \sin 2\phi_1 & \sin 2\phi_2 \end{vmatrix} = 2 \sin \phi_1 \sin \phi_2 (\cos \phi_2 - \cos \phi_1) = 0,$$

which implies  $\cos \phi_1 = \cos \phi_2$ . Hence, we conclude that  $\varepsilon_2 = \bar{\varepsilon}_1$  and that  $\gamma_1 = \gamma_2 = 1/2$ . Put  $\phi = \phi_1$ . Then, we take the real part of (2.3) to obtain  $\cos \phi = p_1/2$ . Thus we have seen that  $P$  has the required form. The remaining part can be easily shown by solving the differential equations  $zf'(z)/f(z) = 1 + zg''(z)/g'(z) = P(z)$ . The proof is now complete.  $\square$

**Remark 2.4.** Under the conditions  $p_1, p_2 \in \mathbb{R}$ , as was seen in the proof,  $D_2 = 0$  if and only if either  $p_2 = p_1^2 - 2$  or  $p_2 = 2$ . The latter case occurs precisely when

$$P(z) = (1 - \gamma) \frac{1 + z}{1 - z} + \gamma \frac{1 - z}{1 + z} = 1 + 2(1 - 2\gamma)z + 2z^2 + \cdots$$

for some constant  $\gamma \in [0, 1]$ .

By making use of the Carathéodory-Toeplitz theorem, Libera and Złotkiewicz [7] showed the following lemma.

**Lemma 2.5.** Let  $-2 \leq p \leq 2$  and  $p_2, p_3 \in \mathbb{C}$ . There exists a function  $P \in \mathcal{P}$  with  $P(z) = 1 + pz + p_2 z^2 + p_3 z^3 + \cdots$  if and only if

$$(2.5) \quad 2p_2 = p^2 + x(4 - p^2).$$

and

$$(2.6) \quad 4p_3 = p^3 + 2(4 - p^2)px - p(4 - p^2)x^2 + 2(4 - p^2)(1 - |x|^2)y$$

for some  $x, y \in \mathbb{C}$  with  $|x| \leq 1$  and  $|y| \leq 1$ .

**Remark 2.6.** It is worth observing the following simple fact. When (2.5) holds with  $x = -1$ , one has the relation  $p_2 = p^2 - 2$  so that the assumption in Lemma 2.3 is satisfied. Therefore, the form of  $P$  can be described by Lemma 2.3 in this case.

For given real numbers  $a, b, c$ , the quantity

$$(2.7) \quad Y(a, b, c) = \max_{z \in \mathbb{D}} (|a + bz + cz^2| + 1 - |z|^2)$$

was used in [8]. It is also helpful in the current study.

**Lemma 2.7.** Let  $a, b, c \in \mathbb{R}$  with  $a \geq 0$  and  $c \geq 0$ . Then

$$Y(a, b, c) = \begin{cases} a + |b| + c & \text{if } |b| \geq 2(1 - c), \\ 1 + a + \frac{b^2}{4(1 - c)} & \text{if } |b| \leq 2(1 - c). \end{cases}$$

The maximum in the definition of  $Y(a, b, c)$  is attained at  $z = \pm 1$  in the first case according as  $b = \pm |b|$ .

When  $|b| = 2(1 - c) = 0$ , we set  $b^2/4(1 - c)$  to be 0 in the above. The lemma can easily be verified by the fact that

$$Y(a, b, c) = \max_{0 \leq r \leq 1} (a + |b|r + cr^2 + 1 - r^2)$$

in this case. See the proof of Proposition 6 in [8] for details.

The next elementary result is helpful to show Lemma 2.9 below. We recall that the discriminant  $\Delta_P$  of the real quadratic polynomial  $P(t) = \alpha + 2\beta t + \gamma t^2$  is defined to be  $\beta^2 - \alpha\gamma$ . We will allow a degenerated case such as  $\gamma = 0$  to define it. Note that  $P(t) = \gamma(t + \beta/\gamma)^2 - \Delta_P/\gamma \geq 0$  for all  $t \in \mathbb{R}$  if  $\gamma > 0$  and if  $\Delta_P \leq 0$ .

**Lemma 2.8.** Let  $P(t)$  and  $Q(t)$  be (possibly degenerated) real quadratic polynomials. Suppose that  $P > 0$  and  $Q > 0$  on an interval  $I \subset \mathbb{R}$  and that  $\Delta_P > 0$ . If there is a positive constant  $T$  such that

- (i)  $\Delta_Q \geq T^{3/2}\Delta_P$ , and
- (ii)  $TP(t) \geq Q(t)$  for  $t \in I$ ,

then the function  $G(t) = \sqrt{P(t)} - \sqrt{Q(t)}$  is convex on  $I$ .

*Proof.* By (i), we have

$$G''(t) = -\frac{\Delta_P}{P(t)^{3/2}} + \frac{\Delta_Q}{Q(t)^{3/2}} \geq \Delta_P \left[ \left( \frac{T}{Q(t)} \right)^{3/2} - \frac{1}{P(t)^{3/2}} \right].$$

Thus the condition (ii) implies that  $G''(t) \geq 0$  for  $t \in I$ . □

The following technical result will be used in the proof of Theorem 1.1.

**Lemma 2.9.** Let  $u = 6p^2/(4 - p^2)$ ,  $v = 2$ ,  $a = 3p^3/(4 - p^2)$ ,  $b = 5p/2$  and  $c = p/2$  for  $4/3 \leq p \leq \sqrt{2}$  and consider the function

$$F(z) = |u + vz| - |a + bz - cz^2|.$$

Then  $F(z) \leq F(-|z|)$  for  $z \in \overline{\mathbb{D}}$ .

*Proof.* Fix an  $r \in (0, 1]$ . A standard computation yields the expression  $F(re^{i\theta}) = G(t)$ , where  $t = \cos \theta$ ,

$$G(t) = \sqrt{A + 2Bt} - \sqrt{L + 2Mt - Nt^2},$$

and  $A = u^2 + v^2r^2$ ,  $B = uvr$ ,  $L = (a + cr^2)^2 + b^2r^2$ ,  $M = br(a - cr^2)$ ,  $N = 4acr^2$ . To apply Lemma 2.8, we put  $\Delta_1 = B^2$ ,  $\Delta_2 = M^2 + LN$  and  $T = 5p^2/8$  and we will show the two inequalities

$$(2.8) \quad \Delta_2 \geq T^{3/2} \Delta_1$$

and

$$(2.9) \quad T(A + 2Bt) \geq L + 2Mt - Nt^2 \quad \text{for } -1 \leq t \leq 1.$$

We first note that

$$\frac{\Delta_2}{\Delta_1} = \frac{(100 - p^2)[6p^2 + (4 - p^2)r^2]^2}{48^2(4 - p^2)} \geq \frac{(100 - p^2)p^4}{64(4 - p^2)}.$$

Since the last quantity is increasing in  $0 < p < 2(\sqrt{6} - 1)$ , we have  $\Delta_2/\Delta_1 \geq 884/405 > 2$  for  $4/3 \leq p \leq \sqrt{2}$ . On the other hand,  $T = 5p^2/8 \leq 5/4$  and thus  $T^{3/2} \leq 5\sqrt{5}/8 < 2$ . The proof of (2.8) is now completed. Next we consider the quadratic polynomial

$$\begin{aligned} R(t) &= T(A + 2Bt) - (L + 2Mt - Nt^2) \\ &= \frac{p^2[54p^4 - 3(4 - p^2)(20 - p^2)r^2 - (4 - p^2)^2r^4]}{4(4 - p^2)^2} + \frac{5p^2r^3}{2}t + \frac{6p^4r^2}{4 - p^2}t^2. \end{aligned}$$

Then its discriminant is

$$\Delta_R = -\frac{p^4r^2}{16(4 - p^2)^3}H(p^2, r^2),$$

where

$$H(x, y) = 36^2x^3 - 72x(4 - x)(20 - x)y - (100 - x)(4 - x)^2y^2.$$

Since the partial derivative  $H_y(x, y)$  is negative for  $0 < x \leq 2$  and  $y > 0$ , one gets  $H(p^2, r^2) \geq H(p^2, 1)$  for  $4/3 \leq p \leq \sqrt{2}$  and  $0 \leq r \leq 1$ . It is easy to verify that  $H(x, 1) = 1225x^3 + 1620x^2 - 4944x - 1600$  is increasing in  $(4/3)^2 \leq x$ . Hence,  $H(p^2, 1) \geq H((4/3)^2, 1) = 2^6 \cdot 18379/3^6 > 0$  for  $4/3 \leq p \leq \sqrt{2}$ . Therefore, we have proved the inequality  $\Delta_R < 0$  and therefore (2.9).

Convexity of  $G$  implies that  $G(t) \leq \max\{G(1), G(-1)\} = \max\{F(r), F(-r)\}$  for  $t \in [-1, 1]$ . We will show that  $F(r) \leq F(-r)$ . To this end, we first observe that the polynomial  $Q(x) = a + bx - cx^2$  has two roots  $x_-$  and  $x_+$  with  $0 < -x_- < x_+$  because  $c > 0, b > 0$ . Moreover, the inequality  $Q(1) = a + b - c = a + 2p > 0$  implies  $x_+ > 1$ . We note also that for  $x \in \mathbb{R}$ ,  $x_- \leq x \leq x_+$  if and only if  $Q(x) \geq 0$ . In view of  $u > v = 2$ , we now obtain

$$\begin{aligned} F(-r) - F(r) &= [(u - vr) - (u + vr)] - (|Q(-r)| - |Q(r)|) \\ &= -4r - |Q(-r)| + |Q(r)| \\ &= \begin{cases} -4r + 2br & \text{if } r < |x_-|, \\ -4r + 2a - 2cr^2 & \text{if } |x_-| \leq r. \end{cases} \end{aligned}$$

Here, we see that  $-4r + 2br = (5p - 4)r \geq 0$ . We also have  $-4r + 2a - 2cr^2 = [(6 + r^2)p^3 + 4p^2r - 4pr^2 - 16r]/(4 - p^2)$ . By monotonicity in  $p \geq 4/3$ , we observe that  $(6 +$

$r^2)p^3 + 4p^2r - 4pr^2 - 16r \geq 16(24 - 15r - 5r^2)/27 > 0$ . Therefore, we conclude now that  $F(-r) - F(r) \geq 0$  at any event, as required.  $\square$

### 3. PROOF OF THEOREM 1.3

We recall the well-known fact that for an analytic function  $f$  on  $\mathbb{D}$  with  $f(0) = 0$ ,  $f'(0) = 1$ ,  $f \in \mathcal{K}$  if and only if  $\operatorname{Re} [1 + zf''(z)/f'(z)] > 0$  on  $|z| < 1$ . Therefore, for  $f \in \mathcal{K}$ , there is a function  $P \in \mathcal{P}$  such that

$$(3.1) \quad f'(z) + zf''(z) = P(z)f'(z).$$

We write

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n.$$

Equating the coefficients of  $z^n$  in both sides of (3.1) for  $n = 1, 2, 3$ , we obtain

$$(3.2) \quad a_2 = \frac{p_1}{2}, \quad a_3 = \frac{p_1^2 + p_2}{6}, \quad \text{and} \quad a_4 = \frac{p_1^3 + 3p_1p_2 + 2p_3}{24}.$$

For  $f \in \mathcal{K}(p)$ , we have  $p_1 = 2a_2 = p$ . Therefore, by Lemma 2.5, for some  $x \in \overline{\mathbb{D}}$  we have

$$\begin{aligned} |a_3 - a_2| &= \left| \frac{p_1^2 + p_2}{6} - \frac{p_1}{2} \right| = \left| \frac{x(4 - p^2)}{12} - \frac{p(2 - p)}{4} \right| \\ &\leq \frac{4 - p^2}{12} + \frac{p(2 - p)}{4} = \frac{-2p^2 + 3p + 2}{6} = \frac{(2p + 1)(2 - p)}{6}. \end{aligned}$$

Here, equality occurs if  $x = -1$ . Thus (1.5) has been shown. Since  $p \in [0, 2]$ , we have

$$|a_3 - a_2| \leq \frac{-2p^2 + 3p + 2}{6} = \frac{-2(p - 3/4)^2 + 25/8}{6} \leq \frac{25}{48},$$

which proves (1.7). Here, equalities hold simultaneously precisely when  $x = -1$  and  $p = 3/4$ . Now Lemma 2.3 together with Remark 2.6 gives the required form of the extremal function.

Next we show (1.6). By substituting (2.5) and (2.6) into (3.2), we have

$$\begin{aligned} |a_4 - a_3| &= \frac{1}{24} |2p_3 + 3p_1p_2 - 4p_2 + p_1^3 - 4p_1^2| \\ &= \frac{1}{24} \left| (4 - p^2)(1 - |x|^2)y + 3p^3 - 6p^2 + \frac{(4 - p^2)(5p - xp - 4)x}{2} \right| \\ &\leq \frac{(4 - p^2)(1 - |x|^2)}{24} + \frac{1}{24} \left| 6p^2 - 3p^3 + \frac{(4 - p^2)((4 - 5p)x + x^2p)}{2} \right| \\ &\leq \frac{4 - p^2}{24} Y(a, b, c), \end{aligned}$$

where  $Y(a, b, c)$  is given in (2.7) and

$$a = \frac{3p^2}{2 + p}, \quad b = \frac{4 - 5p}{2}, \quad c = \frac{p}{2}.$$



Note here that equalities hold simultaneously in the above for a suitable choice of  $x$  and  $y$ . A simple computation tells us that for  $p \in [0, 2]$ ,  $|b| < 2(1 - c)$  if and only if  $0 < p < 8/7$ . Thus Lemma 2.7 yields

$$Y(a, b, c) = \begin{cases} 1 + \frac{3p^2}{2+p} + \frac{(5p-4)^2}{8(2-p)} = \frac{p^3 + 50p^2 - 64p + 64}{8(4-p^2)}, & \text{if } 0 \leq p \leq \frac{8}{7}, \\ a - b + c = \frac{2(3p^2 + 2p - 2)}{2+p}, & \text{if } 8/7 \leq p \leq 2. \end{cases}$$

Hence, we have the estimate  $|a_4 - a_3| \leq \psi(p)$  for  $f \in \mathcal{K}(p)$ , where

$$(3.3) \quad \psi(p) = \begin{cases} \frac{p^3 + 50p^2 - 64p + 64}{192}, & \text{if } 0 \leq p < \frac{8}{7}, \\ \frac{(2-p)(3p^2 + 2p - 2)}{12} = \frac{-3p^3 + 4p^2 + 6p - 4}{12}, & \text{if } 8/7 \leq p \leq 2. \end{cases}$$

Since  $\psi(p)$  is convex in  $0 \leq p \leq 8/7$ , we see that  $\psi(p) \leq \max\{\psi(0), \psi(8/7)\} = \max\{\frac{1}{3}, \frac{103}{343}\} = 1/3$ . On the other hand, we see that  $\psi(p)$  takes its maximum value in  $8/7 \leq p \leq 2$  at  $p = (4 + \sqrt{70})/9$ , which is  $(35\sqrt{70} - 49)/729 \approx 0.334473 > 1/3$ . Therefore, the maximum of  $\psi(p)$  is taken at  $p = (4 + \sqrt{70})/9$  with the choice  $x = -1$  so that the required form of the extremal function follows from Lemma 2.3 and Remark 2.6.

#### 4. PROOFS OF THEOREMS 1.1 AND 1.2

We begin with the proof of Theorem 1.1.

*Proof of Theorem 1.1.* By Alexander's Theorem (see [2, Theorem 2.12]),  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  is convex if and only if  $zf'(z) = z + \sum_{n=2}^{\infty} n a_n z^n$  is starlike. Thus, by Theorem A, we have

$$(4.1) \quad -1 \leq (n+1)|a_{n+1}| - n|a_n| \leq 1$$

for a convex function  $f$  and equalities occur only when  $f$  is a rotation of  $L_\phi$  given in (1.4) for some  $\phi \in \mathbb{R}$ . In particular,

$$|a_{n+1}| - |a_n| \leq |a_{n+1}| - \frac{n}{n+1}|a_n| \leq \frac{1}{n+1}.$$

Note that the function  $f = L_{\pi/n}$  satisfies  $a_n = 0$ ,  $a_{n+1} = -1/(n+1)$ . Thus we have  $\mathcal{D}_n^+ = 1/(n+1)$ , which proves part (i).

As we noted in Introduction, to compute  $\mathcal{D}_n^-$ , we can restrict the range of  $f$  to  $\mathcal{K}^+ = \bigcup_{0 \leq p \leq 2} \mathcal{K}(p)$ . For  $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$  in  $\mathcal{K}(p)$  with  $0 \leq p \leq 2$ , by (3.2) and (2.5), we have

$$\begin{aligned} |a_2| - |a_3| &= \frac{p_1}{2} - \left| \frac{p_1^2 + p_2}{6} \right| = \frac{p}{2} - \frac{|3p^2 + x(4-p^2)|}{12} \\ &\leq \begin{cases} \frac{p}{2} \leq \frac{1}{2}, & \text{for } 0 \leq p \leq 1, \\ \frac{p}{2} - \left( \frac{p^2}{4} - \frac{4-p^2}{12} \right) \leq \frac{1}{2}, & \text{for } 1 \leq p \leq 2. \end{cases} \end{aligned}$$

We have thus obtained  $\mathcal{D}_2^- = 1/2$  and equality is attained when  $p = 1$ ,  $x = -1$ , which corresponds to the function  $L_\phi$  with  $\phi = \arccos[1/2] = \pi/3$  by Lemma 2.3 and Remark 2.6. Now the proof of part (ii) is complete.

In the proof of Theorem 1.3, we saw that  $|a_4 - a_3| \leq 1/3$  when  $0 \leq p \leq 8/7$ . On the other hand, because  $(-3p^3 + 4p^2 + 6p - 4)/12 - 1/3 = -(3p - 4)(p^2 - 2)/12$ , we see that  $\psi(p) \geq 1/3$  only if  $4/3 \leq p \leq \sqrt{2}$ . Hence, by virtue of Theorem 1.3, we can harmlessly assume that  $4/3 \leq p \leq \sqrt{2}$  to show  $|a_3| - |a_4| \leq 1/3$  for  $f \in \mathcal{K}(p)$ .

By (3.2), (2.5), (2.6), we have

$$\begin{aligned} |a_3| - |a_4| &= \left| \frac{p_1^2 + p_2}{6} \right| - \left| \frac{p_1 p_2}{8} + \frac{p_1^3}{24} + \frac{p_3}{12} \right| \\ &= \frac{(4 - p^2)(1 - |x|^2)}{24} + \frac{4 - p^2}{24} \left( \left| \frac{6p^2}{4 - p^2} + 2x \right| - \left| \frac{(5x - x^2)p}{2} + \frac{3p^3}{4 - p^2} \right| \right). \end{aligned}$$

We now apply Lemma 2.8 to get

$$\begin{aligned} |a_3| - |a_4| &\leq \frac{(4 - p^2)(1 - r^2)}{24} + \frac{4 - p^2}{24} \left( \left( \frac{6p^2}{4 - p^2} - 2r \right) - \left| \frac{(-5r - r^2)p}{2} + \frac{3p^3}{4 - p^2} \right| \right) \\ &= \frac{4 - p^2}{24} \left( -r^2 - 2r + \frac{6p^2}{4 - p^2} + 1 - \frac{p}{2} \left| r^2 + 5r - \frac{6p^2}{4 - p^2} \right| \right) \\ &=: \frac{4 - p^2}{24} \Phi(r). \end{aligned}$$

Let  $r_0$  be the unique positive root of the quadratic polynomial  $P(r) = r^2 + 5r - 6p^2/(4 - p^2)$  in  $r$ . Since  $P(1) = 12(2 - p^2)/(4 - p^2) \geq 0$ , we have  $0 < r_0 \leq 1$ . When  $0 \leq r \leq r_0$ ,

$$\Phi(r) = \left( \frac{p}{2} - 1 \right) r^2 + \left( \frac{5p}{2} - 2 \right) r + 1 + \frac{3p^2}{2 + p},$$

Since the right-hand side is increasing in  $0 \leq r \leq 1$ , we have  $\Phi(r) \leq \Phi(r_0)$  for  $0 \leq r \leq r_0$ . When  $r_0 \leq r \leq 1$ ,

$$\Phi(r) = \left( -\frac{p}{2} - 1 \right) r^2 + \left( -\frac{5p}{2} - 2 \right) r + 1 + \frac{3p^2}{2 + p}.$$

The right-hand side is decreasing in  $0 \leq r \leq 1$  so that  $\Phi(r) \leq \Phi(r_0)$  for  $r_0 \leq r \leq 1$ . Hence, we obtain

$$\begin{aligned} |a_3| - |a_4| &\leq \frac{4 - p^2}{24} \Phi(r_0) = \frac{4 - p^2}{24} \left( -r_0^2 - 2r_0 + \frac{6p^2}{4 - p^2} + 1 \right) \\ &= \frac{4 - p^2}{24} (3r_0 + 1). \end{aligned}$$

The inequality  $(4 - p^2)(3r_0 + 1)/24 \leq 1/3$  is equivalent to  $r_0 \leq (4 + p^2)/3(4 - p^2)$ . This can indeed be verified as

$$P\left(\frac{4 + p^2}{3(4 - p^2)}\right) = \frac{8(2 - p^2)(16 - 5p^2)}{9(4 - p^2)^2} \geq 0.$$

Here, equality holds when  $p = \sqrt{2}$ . By tracing back the above proof, we see that all equalities hold, in addition, when  $x = y = -1$ . Again, by Lemma 2.3 and Remark 2.6,

we see that an extremal function is given as  $L_\phi$  with  $\phi = \arccos[\sqrt{2}/2] = \pi/4$ . Thus the proof of part (iii) has been complete.  $\square$

We finish the note by proving Theorem 1.2.

*Proof of Theorem 1.2.* First, we recall the fact that  $|a_n| \leq 1$  for a convex function  $f(z) = z + a_2 z^2 + \dots$  (see [2, p. 45]). Thus, the first inequality in (4.1) together with this yields

$$|a_n| - |a_{n+1}| = \frac{n|a_n| - (n+1)|a_{n+1}|}{n+1} + \frac{|a_n|}{n+1} \leq \frac{2}{n+1}.$$

Here, we note that equality never holds above in view of the equality cases in Theorem A. Thus the right-hand inequality in the theorem has been proved.

As we noted in Introduction, the function  $L_\phi$  given in (1.4) belongs to the class  $\mathcal{K}$ . Therefore, we have

$$(4.2) \quad \mathcal{D}_n^- \geq \max_{\phi \in \mathbb{R}} \Psi_n(\phi),$$

where

$$\Psi_n(\phi) = \left| \frac{\sin n\phi}{n \sin \phi} \right| - \left| \frac{\sin(n+1)\phi}{(n+1) \sin \phi} \right|.$$

We certainly have  $\Psi_n(\theta_n) = 1/n$ , which implies  $\mathcal{D}_n^- \geq 1/n$ . Here,

$$\theta_n = \frac{\pi}{n+1}.$$

When  $n = 2$  or  $3$ ,  $1/n$  is an extremal value for  $\mathcal{D}_n^-$  as we saw before. However, this is not the case when  $n \geq 4$ . Indeed, we will show that the left derivative  $\Psi'_n(\theta_n-)$  is negative for  $n \geq 4$ , which implies that  $\Psi_n(\theta_n - \delta) > 1/n$  for a small enough  $\delta > 0$ . For  $0 < \phi < \theta_n$ , we have the expression

$$\Psi_n(\phi) = \frac{\sin n\phi}{n \sin \phi} - \frac{\sin(n+1)\phi}{(n+1) \sin \phi}.$$

Therefore, we compute

$$\Phi'_n(\theta_n-) = \frac{1 - \frac{n+1}{n} \cos \theta_n}{\sin \theta_n} = \frac{n+1}{n} \cdot \frac{H(\frac{1}{n+1})}{\sin \theta_n},$$

where

$$H(x) = 1 - x - \cos \pi x.$$

We note that  $H(x)$  is strictly convex in  $0 < x < 1/2$  because  $H''(x) = \pi^2 \cos \pi x > 0$  there. Since  $H(0) = 0$ ,  $H(1/5) = (11 - 5\sqrt{5})/20 = -0.0090\dots < 0$ , the inequality  $H(x) < 0$  holds for  $0 < x \leq 1/5$ . Hence,  $\Phi'_n(\theta_n-) < 0$  for  $n \geq 4$  as required.  $\square$

The above proof showed that

$$\max_{\phi \in \mathbb{R}} \Psi_n(\phi) > \frac{1}{n}, \quad n = 4, 5, 6, \dots$$

It seems difficult to find the exact value of this maximum. For instance, a numerical computation tells us that the maximum of  $\Psi_4(\phi)$  is attained at  $\phi_4 \approx 0.19834315\pi$  and its value is approximately 0.250049846. See Figure 1, which was generated by Mathematica Ver. 10.2, for the graph of the function  $\Psi_4(\phi)$ . The first peak is located slightly left to

the bending point  $\phi = \theta_4$ . It might be meaningful to ask whether equality holds or not in (4.2) for  $n \geq 4$ .

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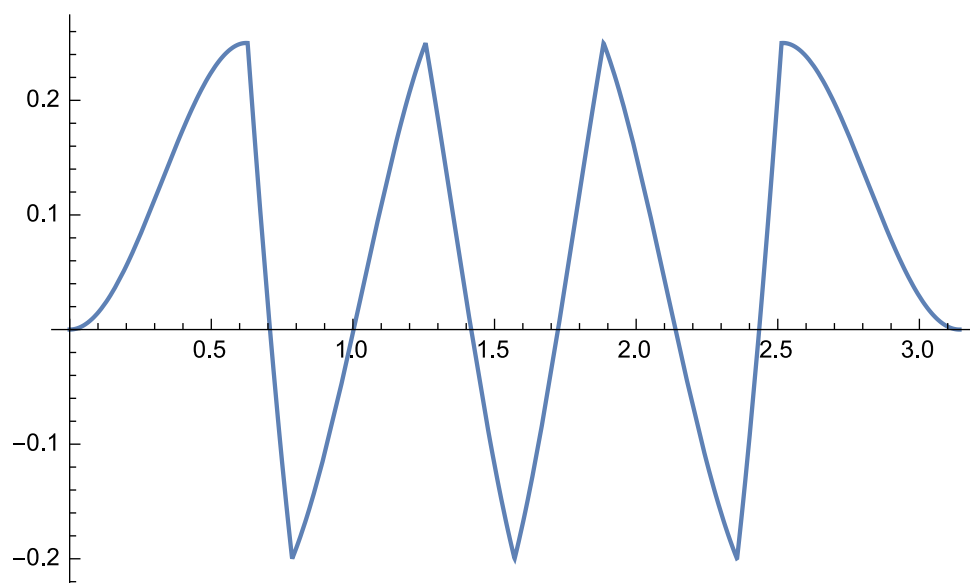


FIGURE 1. The graph of  $\Psi_4(\phi)$ .

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